M3S3/M4S3 - EXERCISES 1

1. *CONTINUITY OF MEASURES*
Prove that if $E_1 \subset E_2 \subset E_3 \subset \ldots$ is an increasing sequence of sets, with

$$E \equiv \lim_{n\to\infty} E_n \equiv \bigcup_{i=1}^{\infty} E_i \qquad \text{then} \qquad P(E) = P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

Use the result that, for $n \geq 1$

$$E_{n+1} \equiv E_n \cup \left(E_{n+1} \cap E_n'\right)$$

where the two sets on the RHS are disjoint. Also prove the result for decreasing sets: if $E_1 \supset E_2 \supset E_3 \supset \ldots$ is a decreasing sequence of sets, with

$$E \equiv \lim_{n\to\infty} E_n \equiv \bigcap_{i=1}^{\infty} E_i \qquad \text{then} \qquad P(E) = P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

Here P is a probability measure, but the results hold for an arbitrary measure ν.

2. *PROPERTIES OF INTEGRAL WRT MEASURE.*
Consider a measure space $(\Omega, \mathcal{F}, \nu)$, Borel measurable functions f and g, and measurable sets $E, E_1, E_2 \in \mathcal{F}$. Using the definition of integral with respect to measure based on simple functions and the supremum definition, prove the following elementary properties:

(a) If f is non-negative such that

$$\int_E f \, d\nu < \infty,$$

and $A \equiv \{\omega \in \Omega : \omega \in E \text{ and } f(\omega) = \infty\}$, then $\nu(A) = 0$.

(b) If $\nu(E) < \infty$, and $m \leq f(\omega) \leq M$ for some real constants $m \leq M$, then f is integrable and

$$m\nu(E) \leq \int_E f \, d\nu \leq M\nu(E)$$

(c) If $\nu(E) = 0$, then

$$\int_E f \, d\nu = 0.$$

(d) If f is non-negative, and

$$\int_E f \, d\nu = 0$$

then $f = 0$ almost everywhere (a.e.) on E, that is, the set $B \equiv \{\omega \in E : f(\omega) > 0\}$ has measure zero under ν.

(e) If f is non-negative, and $E_1 \subseteq E_2$, then

$$\int_{E_1} f \, d\nu \leq \int_{E_2} f \, d\nu.$$

(f) If f and g are non-negative with $f \leq g$ almost everywhere (a.e.) on E, then

$$\int_E f \, d\nu \leq \int_E g \, d\nu.$$