M3S3/M4S3
ASSESSED COURSEWORK 1

Deadline: Friday 26th November 12.00pm
Please hand in to the General Office

1. The Lindeberg-Feller Central Limit Theorem for independent, but not necessarily identically distributed, zero mean random variables can be re-stated as follows: Let $\{X_{nj}, j = 1, 2, ..., n\}$, for $n = 1, 2, 3, ...$, define a triangular lattice, with $E[X_{nj}] = 0$ for all n, j, and suppose $Var[X_{nj}] = \sigma_{nj}^2$. Let

$$T_n = \sum_{j=1}^{n} X_{nj} \tag{1}$$

and let $v_n^2 = Var[T_n] = \sum\limits_{j=1}^{n} \sigma_{nj}^2$. Then

$$\frac{T_n}{v_n} \xrightarrow{\mathfrak{L}} Z \sim N(0,1) \qquad \text{so that} \qquad T_n \sim AN\left(0, v_n^2\right)$$

if

$$\frac{1}{v_n^2} \sum_{j=1}^{n} E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] \to 0 \qquad \text{as } n \to \infty \tag{2}$$

for every $\varepsilon > 0$, where (2) is the Lindeberg condition in a slightly different form.

(a) Show that the Lindeberg-Feller Theorem applies for the triangular lattice defined by the independent random variables

$$X_{nj} = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ \\ 1 & \text{with probability } \frac{1}{2} \end{cases} \qquad j = 1, 2, ..., n \text{ and } n = 1, 2, 3...$$

and find an asymptotic normal approximation for T_n as defined by (1).

[4 MARKS]

(b) Does the Lindeberg-Feller Theorem apply for the triangular lattice defined by the following independent random variables ?

$$X_{nj} = \begin{cases} -j & \text{with probability } \frac{1}{2} \\ \\ j & \text{with probability } \frac{1}{2} \end{cases} \qquad j = 1, 2, ..., n \text{ and } n = 1, 2, 3...$$

If so, find an asymptotic normal distribution for T_n as defined by (1).

[6 MARKS]

2.. (i) Suppose $\{X_n\}$ are a sequence of random variables with $X_n \sim Poisson(\lambda)$ for real parameter $\lambda > 0$. Find an asymptotic normal approximation for the distribution of random variable

$$Y_n = \overline{X}_n \exp\left\{-\overline{X}_n\right\}$$

which can be used as the basis for a hypothesis test about $P\left[X = 1\right] = \lambda e^{-\lambda}$.

[5 MARKS]

(ii) The success parameter, θ, in a Bernoulli experiment is to be estimated from a sequence of independent trials. For which (if any) values of θ does experimental procedure A yield an estimator with asymptotic normal distribution having a lower variance than procedure B ?

PROCEDURE A: n trials are carried out and the number X of successes is recorded: estimator is

$$\widehat{\theta}_A = \frac{X}{n}$$

PROCEDURE B: n paired trials are carried out and the number Y of pairs of successes is recorded: estimator is

$$\widehat{\theta}_B = \sqrt{\frac{Y}{n}}$$

(the probability of a pair of successes on any paired trial in procedure B is $\phi = \theta^2$; the maximum likelihood estimator of ϕ is $\widehat{\phi} = Y/n$, and thus by invariance $\theta = \sqrt{\phi}$ is estimated by $\sqrt{\widehat{\phi}} = \sqrt{Y/n}$).

[5 MARKS]