M3S3/M4S3 - EXERCISES 4

BAYESIAN CALCULATIONS

1. In the identity for scalar x

$$A(x-a)^2 + B(x-b)^2 = C(x-c)^2 + d$$

find the constants c, C and d in terms of quantities A, B, a, b. Hence show that in the standard Bayesian calculation for data $X_1, \ldots X_n$ iid from model with likelihood/prior components

$$f_{X|\mu}(X|\mu) \equiv N(\mu, 1)$$

$$p_\mu(\mu) \equiv N(\theta, \tau^2)$$

with (θ, τ^2) as fixed constants (known as *hyperparameters*), the posterior distribution for μ given $x_1, \ldots, x_n$ is also Normal.

Find a similar identity when $\boldsymbol{x}$ is a $d \times 1$ vector; that is, find an expression equating to

$$(\boldsymbol{x}-\boldsymbol{a})^\mathsf{T} A(\boldsymbol{x}-\boldsymbol{a}) + (\boldsymbol{x}-\boldsymbol{b})^\mathsf{T} B(\boldsymbol{x}-\boldsymbol{b})$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are $d \times 1$ vectors and A and B are $d \times d$ matrices.

2. Suppose that n i.i.d. random variables, with probability model $f_{X|\boldsymbol{\theta}}$ in parameters $\boldsymbol{\theta}$, are partitioned into two blocks $\underset{\sim}{X} = (\underset{\sim}{X}_1, \underset{\sim}{X}_2)^\mathsf{T}$, where $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ are $n_1 \times 1$ and $n_2 \times 1$ vectors respectively. Show that the posterior distribution for $\boldsymbol{\theta}$ has the representation

$$p_{\boldsymbol{\theta}|\underset{\sim}{X}}(\boldsymbol{\theta}|\underset{\sim}{x}) = \frac{L_{n_2}(\boldsymbol{\theta}) p_{\boldsymbol{\theta}|\underset{\sim}{X}_1}(\boldsymbol{\theta}|\underset{\sim}{x}_1)}{\displaystyle\int L_{n_2}(\boldsymbol{\theta}) p_{\boldsymbol{\theta}|\underset{\sim}{X}_1}(\boldsymbol{\theta}|\underset{\sim}{x}_1)\, d\boldsymbol{\theta}}$$

where $L_{n_2}(\boldsymbol{\theta})$ is the likelihood for data $\underset{\sim}{X}_2$ alone, and $p_{\boldsymbol{\theta}|\underset{\sim}{X}_1}(\boldsymbol{\theta}|\underset{\sim}{x}_1)$ is the posterior distribution for $\boldsymbol{\theta}$ in light of data $\underset{\sim}{X}_2 = \underset{\sim}{x}_2$ alone.

3. Find the **posterior predictive** density, $f_{\underset{\sim}{X}^\star|\underset{\sim}{X}}$ for potential future data $\underset{\sim}{X}^\star$ (a vector of $n^\star$ values) given $\underset{\sim}{X} = \underset{\sim}{x}$ using the definition

$$f_{\underset{\sim}{X}^\star|\underset{\sim}{X}}(\underset{\sim}{x}^\star|\underset{\sim}{x}) = \int f_{\underset{\sim}{X}^\star|\boldsymbol{\theta}}(\underset{\sim}{x}^\star|\boldsymbol{\theta}) p_{\boldsymbol{\theta}|\underset{\sim}{X}}(\boldsymbol{\theta}|\underset{\sim}{x})\, d\boldsymbol{\theta} = \int \left\{ \prod_{i=1}^{n^\star} f_{X|\boldsymbol{\theta}}(x_i^\star|\boldsymbol{\theta}) \right\} p_{\boldsymbol{\theta}|\underset{\sim}{X}}(\boldsymbol{\theta}|\underset{\sim}{x})\, d\boldsymbol{\theta}$$

and $p_{\boldsymbol{\theta}|\underset{\sim}{X}}(\boldsymbol{\theta}|\underset{\sim}{x})$ is the usual posterior distribution, if the model is specified as follows:

(i)

$$\begin{aligned} \text{Likelihood} &: X_i|\lambda \sim Poisson(\lambda) \\ \text{Prior} &: \lambda \sim Gamma(\alpha, \beta) \end{aligned}$$

(ii) The model specifies

$$\begin{aligned} \text{Likelihood} &: X_i|\theta \sim Binomial(K, \theta) \\ \text{Prior} &: \theta \sim Beta(\alpha, \beta) \end{aligned}$$

for fixed non-negative integer K.

4. The *exponential family* of distributions includes probability models with mass/density function of the form

$$f_{\boldsymbol{X}|\boldsymbol{\theta}}(\boldsymbol{x}|\boldsymbol{\theta}) = \exp\left\{\boldsymbol{t}(\boldsymbol{x})^{\mathsf{T}}\boldsymbol{a}(\boldsymbol{\theta}) + c(\boldsymbol{\theta}) + d(\boldsymbol{x})\right\}$$

where $\boldsymbol{t}(\boldsymbol{x})$ is a vector function of the datum $\boldsymbol{x}$.

Find the form of an appropriate conjugate prior distribution for $\boldsymbol{\theta}$, and the resulting posterior distribution.

5. Suppose that $X_1, \ldots, X_n$ are iid random variables having a Normal distribution, that is, $X_i \sim N(\mu, \phi)$, so that $Var[X_i] = \phi$, for $i = 1, \ldots, n$.

Assuming a conjugate prior specification for (μ, ϕ) with decomposition

$$p_{\mu,\phi}(\mu, \phi) = p_{\phi}(\phi)p_{\mu|\phi}(\mu|\phi)$$

find the marginal posterior density for μ.

6. *Jeffreys' Prior* for a parameter vector $\boldsymbol{\theta}$ in a probability model is defined by

$$p_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \propto |I(\boldsymbol{\theta})|^{1/2}$$

where I is the Fisher information for $\boldsymbol{\theta}$.

(i) Find Jeffreys' Prior for parameter $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\theta})$ that is a reparameterization of $\boldsymbol{\theta}$.

(ii) Find Jeffreys' Prior if the assumed probability model is $N(\mu, \sigma^2)$.