M3/M4S3 STATISTICAL THEORY II
INTEGRAL WITH RESPECT TO MEASURE

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, and ψ be a **non-negative** simple function, $\psi : \Omega \longrightarrow \mathbb{R}^{\star}$, that is, for $\omega \in \Omega$,

$$\psi(\omega) = \sum_{i=1}^{k} a_i I_{A_i}(\omega)$$

for real constants $a_1, ..., a_k \geq 0$ and measurable sets $A_1, ..., A_k \in \mathcal{F}$, for some $k = 1, 2, 3, ...$, where $I_A(\omega)$ is the indicator function for set A.

(I) The **integral of ψ with respect to ν** is denoted

$$\int_{\Omega} \psi \, d\nu$$

and defined by

$$\int_{\Omega} \psi \, d\nu = \sum_{i=1}^{k} a_i \nu(A_i).$$

(II) Now suppose that f is a **non-negative** (Borel) measurable function, and let $\mathcal{S}_f$ be the set of all non-negative **simple** functions defined by

$$\mathcal{S}_f \equiv \{\psi : \psi(\omega) \leq f(\omega), \forall \omega \in \Omega\}.$$

Then the integral of f with respect to ν is defined by

$$\int_{\Omega} f \, d\nu = \sup_{\psi \in \mathcal{S}_f} \int_{\Omega} \psi \, d\nu$$

that is, the **supremum** (least upper bound) over all possible choices of k, $a_1, ..., a_k \in \mathbb{R}^+$ and $A_1, ..., A_k \in \mathcal{F}$ such that, for all $\omega \in \Omega$,

$$\psi(\omega) = \sum_{i=1}^{k} a_i I_{A_i}(\omega) \leq f(\omega)$$

We refer to this as the **Supremum Definition**.

(III) Finally, suppose that f is an **arbitrary** measurable function defined on Ω. Then, using the max/min functions

$$f^+(\omega) = \max\{f(\omega), 0\} \qquad f^-(\omega) = \max\{-f(\omega), 0\}$$

so that

$$f(\omega) = f^+(\omega) - f^-(\omega),$$

we define the integral of f with respect to ν by

$$\int_{\Omega} f \, d\nu = \int_{\Omega} f^+ \, d\nu - \int_{\Omega} f^- \, d\nu.$$

where the two integrals on the right hand side are integrals of non-negative functions, and thus given by the supremum definition above.

NOTES

(i) In (III) above, it might be that at least one of the two integrals

$$\int_\Omega f^+ \, d\nu \qquad \int_\Omega f^- \, d\nu.$$

is not finite. If precisely one is finite, we say that

$$\int_\Omega f^+ \, d\nu = \infty.$$

and that the integral of f *exists*. If both are finite, we say that the integral of f *exists and is finite*, and f is *integrable with respect to* ν. If neither is finite, then we say that the integral of f *does not exist*, and f is *not-integrable*.

(ii) For $E \subset \Omega$, we can also define

$$\int_E f \, d\nu = \int_E I_E f \, d\nu$$

(iii) All of the following pieces of notation are equivalent and used in the literature:

$$\int f \, d\nu \qquad \int_\Omega f \, d\nu \qquad \int f(\omega) \, d\nu \qquad \int f(\omega) \, d\nu(\omega) \qquad \int f(\omega) \, \nu(d\omega)$$

(iv) Results from the previous handout (Theorems 1.6 and 1.7) show that measurable functions have representations as limits of sequences of simple functions. Other results (Theorems 1.1-1.5) show that measurability is preserved under composition, and also under limit behaviour.

Consider a non-negative measurable function f. Then by Theorem 1.6

$$f = \lim_{n \longrightarrow \infty} \psi_n$$

for a sequence of non-negative simple functions $\psi_1, \psi_2, \ldots$ with $0 \leq \psi_n(\omega) \leq f(\omega)$, for all n and for all $\omega \in \Omega$. Then it can be shown

$$\lim_{n \longrightarrow \infty} \int \psi_n \, d\nu = \int \lim_{n \longrightarrow \infty} \psi_n \, d\nu = \int f \, d\nu$$