M3/M4S3 STATISTICAL THEORY II
LIMITS FOR REAL FUNCTIONS

Definition : Limits
Let f be a real-valued function of real argument x.

- Limit as $x \longrightarrow \infty$:
$$f(x) \longrightarrow a \quad \text{as} \quad x \longrightarrow \infty$$
or
$$\lim_{x \longrightarrow \infty} f(x) = a$$
if, for all $\varepsilon > 0$, $\exists M = M(\varepsilon)$ such that $|f(x) - a| < \varepsilon$, $\forall\ x > M$

- Limit as $x \longrightarrow x_0^{\pm}$:
$$f(x) \longrightarrow a \quad \text{as} \quad x \longrightarrow x_0^{\pm}$$
or
$$\lim_{x \longrightarrow x_0^{\pm}} f(x) = a$$
if, for all $\varepsilon > 0$, $\exists\ \delta$ such that $|f(x) - a| < \varepsilon$, $\forall\ x_0 < x < x_0 + \delta$ (or, respectively $x_0 - \delta < x < x_0$).

- Limit as $x \longrightarrow x_0$:
$$f(x) \longrightarrow a \quad \text{as} \quad x \longrightarrow x_0$$
or
$$\lim_{x \longrightarrow x_0} f(x) = a$$
if
$$\lim_{x \longrightarrow x_0^+} f(x) = \lim_{x \longrightarrow x_0^-} f(x) = a.$$

Definition : Order Notation
Let $x \longrightarrow x_0$. Then write

$$\begin{aligned}
f(x) \sim g(x) \quad & \text{if} \quad \frac{f(x)}{g(x)} \longrightarrow 1 \quad \text{as} \quad x \longrightarrow x_0 \\
f(x) = o(g(x)) \quad & \text{if} \quad \frac{f(x)}{g(x)} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow x_0 \\
f(x) = O(g(x)) \quad & \text{if} \quad \frac{f(x)}{g(x)} \longrightarrow b \quad \text{as} \quad x \longrightarrow x_0
\end{aligned}$$

Definition : Continuity
Function $f(x)$ is continuous at x_0 if
$$\lim_{x \longrightarrow x_0^+} f(x) = \lim_{x \longrightarrow x_0^-} f(x) = f(x_0)$$
and all limits exist.

Definition : Maximum and Minimum functions
For real-valued functions f and g of $x \in \mathbb{R}$,

$$f(x) \wedge g(x) = \min\{f(x), g(x)\} \qquad f(x) \vee g(x) = \max\{f(x), g(x)\}$$

Definition : Positive and Negative Part functions
For real-valued functions f of $x \in \mathbb{R}$,

$$f^{+}(x) = f(x) \vee 0 = \max\{f(x), 0\} \qquad f^{-}(x) = -f(x) \vee 0 = \max\{-f(x), 0\}$$

so that $f^{+}(x) \geq 0$ and $f^{-}(x) \geq 0$ for all x, and

$$f(x) = f^{+}(x) - f^{-}(x) \qquad |f(x)| = f^{+}(x) + f^{-}(x).$$

EXTREMUM LIMITS FOR SEQUENCES

Definition : Supremum and Infimum
A set of real values S is **bounded above (bounded below)** if there exists a real number a (b) such that, for all $x \in S$, $x \leq a$ ($x \geq b$). The quantity a (b) is an **upper bound (lower bound)**. A real value a_L (b_U) is a **least upper bound** (**greatest lower bound**) if it is an upper bound (a lower bound) of S, and no other upper (lower) bound is smaller (larger) than a_L (b_U). We write

$$a_L = \sup S \qquad b_U = \inf S$$

for the a_L, the **supremum**, and b_U, the **infimum** of S.

If S comprises a sequence of elements $\{x_n\}$, then we can write

$$a_L = \sup_{x_n \in S} x_n \equiv \sup_n x_n \qquad b_U = \inf_{x_n \in S} x_n \equiv \inf_n x_n.$$

A sequence that is both bounded above and bounded below is termed **bounded**.

NOTE : Any bounded, monotone real sequence is **convergent**.

Definition : Limit Superior and Limit Inferior
Suppose that $\{x_n\}$ is a bounded real sequence. Define sequences $\{y_k\}$ and $\{z_k\}$ by

$$y_k = \inf_{n \geq k} x_n \qquad z_k = \sup_{n \geq k} x_n$$

Then $\{y_k\}$ is a bounded non-decreasing sequence and $\{z_k\}$ is a bounded non-increasing sequence, and

$$\lim_{k \to \infty} y_k = \sup_k y_k \qquad \text{and} \qquad \lim_{k \to \infty} z_k = \inf_k z_k.$$

We define the **limit superior** (or **upper** limit, or lim sup) and the **limit inferior** (or **lower** limit, or lim inf) by

$$\limsup x_n = \lim_{k \to \infty} \sup_{n \geq k} x_n = \inf_k \sup_{n \geq k} x_n = \overline{\lim}\, x_n$$

$$\liminf x_n = \lim_{k \to \infty} \inf_{n \geq k} x_n = \sup_k \inf_{n \geq k} x_n = \underline{\lim}\, x_n$$

Then we have $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$ and $\lim x_n = x$ if and only if $\underline{\lim}\, x_n = x = \overline{\lim}\, x_n$.