M3/M4S3 STATISTICAL THEORY II
MEASURABLE FUNCTIONS

The real-valued function f defined with domain $E \subset \Omega$, for measurable space $(\Omega, \mathcal{F})$, is **Borel measurable** with respect to $\mathcal{F}$ if the inverse image of set B, defined as

$$f^{-1}(B) \equiv \{\omega \in E : f(\omega) \in B\}$$

is an element of σ-algebra $\mathcal{F}$, for all Borel sets B of $\mathbb{R}$ (strictly, of the *extended* real number system $\mathbb{R}^*$, including $\pm\infty$ as elements). The following conditions are each necessary and sufficient for f to be measurable

(a) $f^{-1}(A) \in \mathcal{F}$ for all open sets $A \subset \mathbb{R}^*$,

(b) $f^{-1}([-\infty, x)) \in \mathcal{F}$ for all $x \in \mathbb{R}^*$,

(c) $f^{-1}([-\infty, x]) \in \mathcal{F}$ for all $x \in \mathbb{R}^*$,

(d) $f^{-1}([x, \infty]) \in \mathcal{F}$ for all $x \in \mathbb{R}^*$,

(e) $f^{-1}((x, \infty]) \in \mathcal{F}$ for all $x \in \mathbb{R}^*$.

NOTES:

(i) The **Borel σ-algebra** in $\mathbb{R}$, $\mathcal{B}$, is the smallest (or **minimal**) σ-algebra containing all **open sets** (that is, essentially, sets of the form

$$(a, b) \qquad \text{or} \qquad [a, b]'$$

for $a < b \in \mathbb{R}$) which are known as the **Borel sets** in $\mathbb{R}$.

(ii) It is possible to extend this definition to a general **topological space** Ω equipped with a **topology**, that is, a collection, $\mathcal{T}$, of sets in Ω that (I) $\mathcal{T}$ contains $\emptyset$ and Ω, (II) $\mathcal{T}$ is closed under finite intersection, and (III) if $\mathcal{A}$ is a sub-collection of $\mathcal{T}$, $\mathcal{A} \subset \mathcal{T}$, and $A_1, A_2, A_3, ... \in \mathcal{A}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{T}.$$

In this context, it is possible to define a general Borel σ-algebra on Ω; the **open sets** are the elements $T_1, T_2, T_3, ...$ of the topology $\mathcal{T}$, and the Borel sets are the elements of the smallest σ-algebra generated by $\mathcal{T}$, $\sigma(\mathcal{T})$. However, we will not be studying general toplogical spaces; we shall restrict attention to $\mathbb{R}$, and thus refer to **the** Borel sets and **the** Borel σ-algebra, meaning the Borel sets/σ-algebra defined on $\mathbb{R}$.

(iii) Strictly, a function f is a **Borel function** if, for $B \in \mathcal{B}$, $f^{-1}(B) \in \sigma(\mathcal{T})$; however, we will generally consider measure spaces $(\Omega, \mathcal{F})$ and say that f is a **Borel function** if it is Borel measurable, as defined in the first paragraph above.

Example Consider Lebesgue measure, m, defined for real numbers $a < b$ (on the Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}$) by

$$m([a,b]) = m((a,b)) = m((a,b]) = m([a,b)) = b - a.$$

Suppose f is an increasing function on $\mathbb{R}$. Then the set $A \equiv f^{-1}([-\infty, x])$ is an interval in $\mathbb{R}$, and thus f is measurable with respect to Lebesgue measure, as the measure of A, $m(A)$, is well-defined. Now consider the function g defined by $g(x) = x$ for $x \in \mathbb{R}$. This function is measurable with respect to Lebesgue measure (on $\mathcal{B}$), as it is increasing. However, consider the σ-algebra, $\mathcal{Z}$, generated by the sets $\{\emptyset, (-\infty, 0], (0, \infty), \mathbb{R}\}$.Then

$$g^{-1}((-\infty, 1]) \notin \mathcal{Z}$$

so g is not measurable on $\mathcal{Z}$.

RESULTS FOR MEASURABLE FUNCTIONS

Theorem 1.1 ***MEASURABILITY UNDER COMPOSITION***
Let g_1 and g_2 be measurable functions on $E \subset \Omega$ with ranges in $\mathbb{R}^*$. Let f be a Borel function from $\mathbb{R}^* \times \mathbb{R}^*$ into $\mathbb{R}^*$. Then the composite function h, defined on E by

$$h(\omega) = f(g_1(\omega_1), g_2(\omega_2))$$

is measurable.

Proof. *The function $g = (g_1, g_2)$ has domain E and range $\mathbb{R}^* \times \mathbb{R}^*$, and is measurable as g_1 and g_2 are measurable, and denote $h = f \circ g$ (the operator $\circ$ indicates composition, i.e.*

$$h(\omega_1, \omega_2) = (f \circ g)(\omega_1, \omega_2) \qquad \textit{if} \qquad h(\omega_1, \omega_2) = f(g(\omega_1, \omega_2)) = f(g_1(\omega_1), g_2(\omega_2)).$$

If $B \in \mathcal{B}$, then $f^{-1}(B)$ is a Borel set as f is a Borel function. Thus the inverse image under h,

$$h^{-1}(B) = g^{-1}\left(f^{-1}(B)\right)$$

is measurable as g_1 and g_2, and hence g, are measurable.

Corollary If g is a measurable function from E into $\mathbb{R}^*$, and f is a continuous function from $\mathbb{R}^*$ into $\mathbb{R}^*$, then $h = f \circ g$ is measurable.

Theorem 1.2 ***MEASURABILITY UNDER ELEMENTARY OPERATIONS***
Let g_1 and g_2 be measurable functions defined on $E \subset \Omega$ into $\mathbb{R}^*$, and let c be any real number. Then all of the following composite and other related functions are measurable

$$g_1 + g_2, g_1 + c, g_1 g_2, c g_1, g_1/g_2, |g_1|^c, g_1 \vee g_2, g_1 \wedge g_2, g_1^+, g_1^-.$$

Proof. In each case, we examine the domain of the composite function to ensure measurability in the Borel σ-algebra. Consider $g_1 + g_2$; this is not defined on the set

$$\{\omega : g_1(\omega) = -g_2(\omega) = \pm\infty\}$$

(as $\infty \pm \infty$ is not defined), but this set is measurable, and so is the domain of $g_1 + g_2$. Let $f(x_1, x_2) = x_1 + x_2$ be a continuous function defined on $\mathbb{R}^* \times \mathbb{R}^*$. Then, by Theorem 1.1 and its corollary, $g_1 + g_2$ is measurable. Taking $g_2 = c$ proves that $g_1 + c$ is measurable.

The function $g_1 g_2$ is defined everywhere on E; it's measurability follows from Theorem 1.1, setting $f(x_1, x_2) = x_1 x_2$. Setting $g_2 = c$ proves that cg_1 is measurable.

The function g_1/g_2 is defined everywhere except on the union of sets

$$\{\omega : g_1(\omega) = g_2(\omega) = 0\} \cup \{\omega : \pm g_1(\omega) = \pm g_2(\omega) = \infty\}$$

Similarly, if $c = 0$, $|g_1|^c$ is defined except on

$$\{\omega : g_1(\omega) = \pm\infty\};$$

if $c < 0$, it is defined except on

$$\{\omega : g_1(\omega) = 0\}.$$

If $c > 0$, it is defined everywhere. All of these sets are measurable Thus, we consider in turn functions

$$f(x_1, x_2) = x_1/x_2 \qquad f(x) = x^c$$

and use Theorem 1.1.

The functions $g_1 \vee g_2, g_1 \wedge g_2$ are defined everywhere; so we consider functions

$$f(x_1, x_2) = \max\{x_1, x_2\} \qquad f(x_1, x_2) = \min\{x_1, x_2\}$$

and again use Theorem 1.1. Finally, setting $g_2 = 0$ yields the measurability of g_1^+ and g_1^-.

Theorem 1.3 *If g_1 and g_2 are measurable functions on a common domain, then each of the sets*

$$\{\omega : g_1(\omega) < g_2(\omega)\} \qquad \{\omega : g_1(\omega) = g_2(\omega)\} \qquad \{\omega : g_1(\omega) > g_2(\omega)\}$$

is measurable.

Proof. Since g_1 and g_2 are measurable, then $f = g_1 - g_2$ is measurable, and thus the two sets

$$\{\omega : f(\omega) > 0\} \qquad \{\omega : f(\omega) = 0\}$$

are measurable. Since

$$\{\omega : g_1(\omega) < g_2(\omega)\} \equiv \{\omega : f(\omega) > 0\}$$

and

$$\{\omega : g_1(\omega) = g_2(\omega)\} \equiv \{\omega : f(\omega) = 0\} \cup \{\omega : g_1(\omega) = g_2(\omega) = \pm\infty\}$$

then $\{\omega : g_1(\omega) < g_2(\omega)\}$ and $\{\omega : g_1(\omega) = g_2(\omega)\}$ are measurable, and so is

$$\{\omega : g_1(\omega) \leq g_2(\omega)\} \equiv \{\omega : g_1(\omega) < g_2(\omega)\} \cup \{\omega : g_1(\omega) = g_2(\omega)\}.$$

Theorem 1.4 ***MEASURABILITY UNDER LIMIT OPERATIONS***
If $\{g_n\}$ is a sequence of measurable functions, the functions $\sup_n g_n$ and $\inf_n g_n$ are measurable.

Proof. Let $g = \sup_n g_n$. Then for real x, consider

$$g_n^{-1}\left([-\infty, x]\right) \equiv \{\omega : g_n(\omega) \leq x\}$$

and

$$g^{-1}\left([-\infty, x]\right) \equiv \{\omega : g(\omega) \leq x\}.$$

If $g = \sup_n g_n$, then $g_n \leq g$ for all n, and

$$g(\omega) \leq x \Longrightarrow g_n(\omega) \leq x \qquad \text{so that} \qquad \omega \in g^{-1}\left([-\infty, x]\right) \Longrightarrow \omega \in g_n^{-1}\left([-\infty, x]\right)$$

so that

$$g^{-1}\left([-\infty, x]\right) \subseteq g_n^{-1}\left([-\infty, x]\right)$$

for all n. Thus, in fact

$$g^{-1}\left([-\infty, x]\right) = \bigcap_n g_n^{-1}\left([-\infty, x]\right)$$

and hence g is measurable, as the intersection of measurable sets is measurable. The result for $\inf_n$ follows by noting that

$$\inf_n g_n = -\sup_n\left(-g_n\right).$$

Theorem 1.5 ***MEASURABILITY UNDER LIMINF/LIMSUP***
If $\{g_n\}$ is a sequence of measurable functions, the functions $\limsup_n g_n$ and $\liminf_n g_n$ are measurable.

Proof. This follows from Theorem 1.4, as

$$\limsup_n g_n = \inf_k \left\{\sup_{n \geq k} g_n\right\} \qquad \text{and} \qquad \liminf_n g_n = \sup_k \left\{\inf_{n \geq k} g_n\right\}$$

SIMPLE FUNCTIONS AND THEIR CONVERGENCE PROPERTIES.

Definition : Simple Functions
A *simple function*, ψ, is a set function defined on elements ω of sample space Ω by

$$\psi(\omega) = \sum_{i=1}^{k} a_i I_{A_i}(\omega)$$

for real constants $a_1, ..., a_k$ and measurable sets $A_1, ..., A_k$, for some $k = 1, 2, 3, ...$, where $I_A(\omega)$ is the *indicator function*, where

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}.$$

Note that any such simple function, can be re-expressed as a simple function defined for a **partition** of Ω, $E_1, ..., E_l$,

$$\psi(\omega) = \sum_{i=1}^{l} e_i I_{E_i}(\omega)$$

by suitable choice of the constants $e_1, ..., e_k$.

Theorem 1.6 *A non-negative function on Ω is measurable if and only if it is the limit of an increasing sequence of non-negative simple functions.*

Proof. Suppose that g is a nonnegative measurable function. For each positive integer n, define the simple function ψ_n on Ω by

$$\psi_n(\omega) = \frac{m}{2^n} \qquad \text{if } \frac{m}{2^n} \leq g(\omega) < \frac{m+1}{2^n}$$

for $m = 0, 1, 2, ..., 2^n - 1$, and

$$\psi_n(\omega) = n \qquad \text{if } n \leq g(\omega).$$

Then $\{\psi_n\}$ is an increasing sequence of non-negative simple functions. Since

$$|\psi_n(\omega) - g(\omega)| < \frac{1}{2^n} \qquad \text{if } n > g(\omega)$$

and $\psi_n(\omega) = n$ if $g(\omega) = \infty$, then, for all ω,

$$\psi_n(\omega) \to g(\omega)$$

and we have found the sequence required for the result.

Now suppose that g is a limit of an increasing sequence of non-negative simple functions. Then it is measurable by Theorem 1.5.

Theorem 1.7 *A function g defined on Ω is measurable if and only if it is the limit of a sequence of simple functions.*

Proof. Suppose that g is measurable. Then g^+ and g^- are measurable and non-negative, and thus can be represented as limits of simple functions $\{\psi_n^+\}$ and $\{\psi_n^-\}$, by the Theorem 1.6. Consider the sequence of simple functions defined by $\{\psi_n^+ - \psi_n^-\}$; this sequence converges to $g^+ - g^- = g$, and we have the sequence of simple functions required for the result.

Now suppose that g is a limit of a sequence of simple functions. Then it is measurable by Theorem 1.5.