M3S3/M4S3
ASSESSED COURSEWORK 2 : SOLUTIONS

1. The likelihood for $\boldsymbol{\theta}$ is

$$L_n(\boldsymbol{\theta}) \quad = \quad \prod_{i=1}^{n} f_{X|\boldsymbol{\theta}}(x_i|\boldsymbol{\theta}) = \prod_{i=1}^{n} \frac{\alpha}{\lambda^\alpha} x_i^{\alpha-1} \exp\left\{-\left(\frac{x_i}{\lambda}\right)^\alpha\right\} = \frac{\alpha^n}{\lambda^{n\alpha}} t_n^{\alpha-1} \exp\left\{-\left(\frac{1}{\lambda}\right)^\alpha \sum_{i=1}^{n} x_i^\alpha\right\}$$

where $t_n = \prod_{i=1}^{n} x_i$, and hence the log-likelihood is

$$l_n(\boldsymbol{\theta}) \quad = \quad n\log\alpha - n\alpha\log\lambda + (\alpha-1)\log t_n - \frac{1}{\lambda^\alpha}\sum_{i=1}^{n} x_i^\alpha.$$

For the profile likelihood for α, we find the ML estimate of λ as a function of fixed α. Partially differentiating with respect to λ, we obtain

$$\frac{\partial l_n(\boldsymbol{\theta})}{\partial\lambda} = -\frac{n\alpha}{\lambda} + \frac{\alpha}{\lambda^{\alpha+1}}\sum_{i=1}^{n} x_i^\alpha.$$

and equating to zero yields

$$\widehat{\lambda}(\alpha) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^\alpha\right)^{1/\alpha}.$$

Thus the profile likelihood for α is

$$L_P(\alpha) = L_n(\alpha, \widehat{\lambda}(\alpha)) = \frac{(n\alpha)^n t_n^{\alpha-1}}{\left(\sum_{i=1}^{n} x_i^\alpha\right)^n} \exp\{-n\} \qquad \alpha > 0$$

[5 MARKS]

2. Let n_A and n_B denote the numbers of years, and let s_A and s_B be the totals of counts over those years, for roads A and B respectively. Let $\overline{x}_A = s_A/n_A$ and $\overline{x}_B = s_B/n_B$ denote the mean count per year for the two roads, and recall that $\overline{x}_A$ and $\overline{x}_B$ are the ML estimates for the two parameters.

In the original parameterization the likelihood takes the form

$$L_n(\lambda_A, \lambda_B) = c\lambda_A^{s_A}\lambda_B^{s_B}\exp\{-[n_A\lambda_A + n_B\lambda_B]\}$$

where c is a constant that does not depend on the parameters, in fact

$$c = \left(\prod_{i=1}^{n_A} x_{Ai}! \prod_{i=1}^{n_B} x_{Bi}!\right)^{-1}$$

(i) One way to compute the profile likelihood for θ is to reparameterize, say, to

$$\theta = \frac{\lambda_A}{\lambda_B},\ \phi = \lambda_B \qquad \Longrightarrow \qquad \lambda_A = \theta\phi,\ \lambda_B = \phi.$$

Then

$$L_n(\theta, \phi) = c(\theta\phi)^{s_A}\phi^{s_B}\exp\{-[n_A\theta\phi + n_B\phi]\} = c\theta^{s_A}\phi^{s_A+s_B}\exp\{-[n_A\theta + n_B]\phi\}$$

and

$$l_n(\theta,\phi) = \log c + s_A \log\theta + (s_A+s_B)\log\phi - (n_A\theta + n_B)\phi.$$

Taking first derivatives with respect to ϕ yields

$$\frac{\partial l_n(\theta,\phi)}{\partial\phi} = \frac{s_A+s_B}{\phi} - (n_A\theta+n_B)$$

and equating this to zero gives

$$\widehat{\phi}(\theta) = \frac{s_A+s_B}{n_A\theta+n_B}$$

and a profile likelihood

$$L_P(\theta) = L_n(\theta,\widehat{\phi}(\theta)) = c\theta^{s_A}\left(\frac{s_A+s_B}{n_A\theta+n_B}\right)^{s_A+s_B} \exp\{-(s_A+s_B)\}$$

and writing $s = s_A + s_B$, we have

$$L_P(\theta) = cs^s \exp\{-s\}\frac{\theta^{s_A}}{(n_A\theta+n_B)^s}.$$

[3 MARKS]

(ii) There are many possible ways to carry out this test. For example, could test

$$H_0 \;:\; \theta = 1$$

with ϕ **unspecified**, against the general alternative; the likelihood ratio test is most straightforward - let

$$\Lambda_n = \frac{L_n(\widehat{\theta},\widehat{\phi})}{L_n(\theta_0,\widehat{\phi}_0)}$$

where $(\widehat{\theta},\widehat{\phi})$ are the ML estimates under the alternative hypothesis, $\theta_0 = 1$, and $\widehat{\phi}_0$ is the ML estimate of ϕ under the null. Now, the ML estimates under the alternative are available by the principle of **invariance**, that is

$$\widehat{\theta} = \frac{\widehat{\lambda}_A}{\widehat{\lambda}_B} = \frac{\overline{x}_A}{\overline{x}_B} \qquad \widehat{\phi} = \overline{x}_B$$

Under the null, where $\theta = \theta_0 = 1$, the log-likelihood is

$$l_n(1,\phi) = \log c + (s_A+s_B)\log\phi - (n_A+n_B)\phi.$$

and thus

$$\widehat{\phi}_0 = s/n = \overline{x}$$

where $n = n_A + n_B$, that is, $\widehat{\phi}_0$ is the pooled sample estimate of the Poisson parameter common to both samples (as, if $\theta = 1$, $\lambda_A = \lambda_B = \phi$). Hence

$$\Lambda_n \;=\; \frac{L_n(\widehat{\theta},\widehat{\phi})}{L_n(1,\widehat{\phi}_0)} = \frac{(\overline{x}_A/\overline{x}_B)^{s_A}(\overline{x}_B)^s \exp\{-[n_A(\overline{x}_A/\overline{x}_B)+n_B]\overline{x}_B\}}{\overline{x}^s \exp\{-[n_A+n_B]\overline{x}\}} = \frac{(\overline{x}_A)^{s_A}(\overline{x}_B)^{s_B}}{\overline{x}^s}.$$

The LR test is completed by noting that using the standard theory, asymptotically,

$$2\log\Lambda_n = 2(S_A\log\overline{X}_A + S_B\log\overline{X}_B - S\log\overline{X}) \xrightarrow{\mathfrak{L}} \chi^2_1.$$

Here the test statistic is

$$2(s_A\log\overline{x}_A + s_B\log\overline{x}_B - s\log\overline{x}) = 2(12\log 2 + 4\log 1 - 16\log 1.6) = 1.595.$$

From tables, the 95 % quantile from the Chi-squared distribution with 1 degree of freedom is 3.841, so there is no evidence to reject the hypothesis that $\theta = 1$.

Alternatively, could test the null hypothesis

$$H_0 \;:\; \lambda_A = \lambda_B$$

directly using asymptotic normal approximations and the Delta Method directly, using a test statistic based on $\overline{X}_A - \overline{X}_B$. This results in an approximate Z-test. However, it is questionable whether any asymptotic methods would be valid here, where the sample sizes are very small.

[4 MARKS]

3.(i) Again here, the key is **invariance**; we know by elementary results that the ML estimates for π_1 and π_2 are

$$\widehat{\pi}_1 = \frac{x_1}{n_1} \qquad \widehat{\pi}_2 = \frac{x_2}{n_2}$$

and by invariance (or indeed from first principles by writing out the likelihood in full) the ML estimate of θ is

$$\widehat{\theta} = \frac{\widehat{\pi}_1(1-\widehat{\pi}_2)}{\widehat{\pi}_2(1-\widehat{\pi}_1)} = \frac{(x_1/n_1)(1-x_2/n_2)}{(x_2/n_2)(1-x_1/n_1)} = \frac{x_1(n_2-x_2)}{x_2(n_1-x_1)}.$$

[4 MARKS]

(ii) The log-odds ratio is

$$\log\theta = \log\left(\frac{\pi_1}{1-\pi_1}\right) - \log\left(\frac{\pi_2}{1-\pi_2}\right)$$

and thus

$$\log\widehat{\theta}_n = \log\left(\frac{x_1/n_1}{1-x_1/n_1}\right) - \log\left(\frac{x_2/n_2}{1-x_2/n_2}\right)$$

with corresponding estimator

$$\log\left(\frac{X_1/n_1}{1-X_1/n_1}\right) - \log\left(\frac{X_2/n_2}{1-X_2/n_2}\right).$$

As X_1 and X_2 are independent, these two terms are independent random variables, and we can compute the required distribution using the Central Limit Theorem and the Delta Method. We have, by standard methods that

$$\frac{X_1}{n_1} \sim AN\left(\pi_1, \frac{\pi_1(1-\pi_1)}{n_1}\right) \qquad \frac{X_2}{n_2} \sim AN\left(\pi_2, \frac{\pi_2(1-\pi_2)}{n_2}\right)$$

and taking $g(t) = \log(t/(1-t))$ in the Delta method so that

$$\dot{g}(t) = \frac{1}{t(1-t)} \quad \therefore \quad \{\dot{g}(t)\}^2 = \frac{1}{t^2(1-t)^2} \quad \therefore \quad \frac{\{\dot{g}(\pi)\}^2\pi(1-\pi)}{n} = \frac{1}{n\pi(1-\pi)}$$

and we have that

$$\begin{aligned}
\log\left(\frac{X_1/n_1}{1-X_1/n_1}\right) &\sim AN\left(\log\left(\frac{\pi_1}{1-\pi_1}\right), \frac{1}{n_1\pi_1(1-\pi_1)}\right) \\
\log\left(\frac{X_2/n_2}{1-X_2/n_2}\right) &\sim AN\left(\log\left(\frac{\pi_2}{1-\pi_2}\right), \frac{1}{n_2\pi_2(1-\pi_2)}\right).
\end{aligned}$$

Thus for the estimator

$$\log\widehat{\theta}_n \sim AN\left(\log\theta, \frac{1}{n_1\pi_1(1-\pi_1)} + \frac{1}{n_2\pi_2(1-\pi_2)}\right)$$

[4 MARKS]