M3S3/M4S3 - EXERCISES 1

1. Let sequence of random variables $\{X_n\}$, $n = 1, 2, 3, ...$ have density functions

$$f_n(x) = \frac{n}{\pi(1 + n^2x^2)} \qquad -\infty < x < \infty$$

With respect to which modes of convergence does X_n converge as $n \to \infty$?

2. Consider general random variables X and Y.

   (i) Prove **Holder's Inequality**: for constants $p > 1$ and $1/p + 1/q = 1$

$$E[|XY|] \leq \{E[|X|^p]\}^{1/p} \{E[|Y|^q]\}^{1/q}$$

   Use the fact that for real numbers $x, y > 0$ and $t \in (0, 1)$

$$x^t y^{1-t} \leq tx + (1-t)\, y.$$

   (ii) Prove **Minkowski's Inequality**: for $p \geq 1$

$$\{E[|X + Y|^p]\}^{1/p} \leq \{E[|X|^p]\}^{1/p} + \{E[|Y|^p]\}^{1/p}.$$

Hence deduce that if $r_1 > r_2 > 0$ then

$$X_n \stackrel{r_1}{\to} X \Longrightarrow X_n \stackrel{r_2}{\to} X$$

3. Suppose that $X_n \stackrel{a.s.}{\to} X$ and $Y_n \stackrel{a.s.}{\to} Y$ as $n \to \infty$. Show that

$$X_n + Y_n \stackrel{a.s.}{\to} X + Y \tag{1}$$

and

$$X_n Y_n \stackrel{a.s.}{\to} XY \tag{2}$$

Hint: for (1) recall the definition of almost sure convergence and look at the set of $\omega \in \Omega$ such that

$$X_n(\omega) + Y_n(\omega) \nrightarrow X_n(\omega) + Y_n(\omega)$$

Show that (1) holds for convergence in probability and convergence in r^{th} mean, but not convergence in distribution. Does (2) hold for the other modes of convergence ?

4. **Slutsky's Theorems**

   (i) Suppose that $X_n \stackrel{\mathfrak{L}}{\to} X$ and $Y_n \stackrel{p}{\to} c$ for some constant c. Show that

$$X_n Y_n \stackrel{\mathfrak{L}}{\to} cX \qquad \text{and} \qquad X_n/Y_n \stackrel{\mathfrak{L}}{\to} X/c \text{ if } c \neq 0.$$

   (ii) Suppose that $X_n \stackrel{\mathfrak{L}}{\to} 0$ and $Y_n \stackrel{p}{\to} Y$. Let g be a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $g(x, y)$ continuous in y for each x, and that $g(x, y)$ continuous at $x = 0$ for all y. Show that

$$g(X_n, Y_n) \stackrel{p}{\to} g(0, Y)$$

5. Suppose that X_n $(n \geq 2)$ is defined on $\mathbb{X}_n \equiv \{-n, 0, n\}$ by

$$P[X_n = n] = P[X_n = -n] = \frac{1}{2n \log n} \qquad P[X_n = 0] = 1 - \frac{1}{n \log n}.$$

Let $S_n = X_2 + ... + X_n$

$$\frac{S_n}{n} \xrightarrow{p} 0 \qquad \text{but} \qquad \frac{S_n}{n} \overset{a.s.}{\nrightarrow} 0$$

Hint: to prove the first part, show

$$\frac{S_n}{n} \xrightarrow{r} 0$$

for $r = 2$. To prove the second part, assume that $|X_i| \geq i$ for infinitely many i (or *infinitely often*), then choose an i for which this holds; then look at

$$|S_i - S_{i-1}|.$$

6. Let $Y_1, Y_2, ...$ be independent random variables each taking values on the set $\{0, 1, 2, ..., 9\}$ with equal probability. Let

$$X_n = \sum_{i=1}^{n} \frac{Y_i}{10^i}.$$

Use moment generating functions to show that

$$X_n \xrightarrow{\mathfrak{L}} X \sim Uniform\,(0,1).$$

Is it the case that

$$X_n \xrightarrow{a.s.} X \sim Uniform\,(0,1).$$

also ?

CONTINUITY OF MEASURES

Prove the result from lectures: if $E_1 \subseteq E_2 \subseteq E_3 \subseteq ...$ is an increasing sequence of sets, with

$$E \equiv \lim_{n\to\infty} E_n \equiv \bigcup_{i=1}^{\infty} E_i$$

then

$$P(E) = P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

Use the result that, for $n \geq 1$

$$E_{n+1} \equiv E_n \cup \left(E_{n+1} \cap E_n'\right)$$

where the two sets on the RHS are disjoint.

As a corollary, prove the result for decreasing sets: if $E_1 \supseteq E_2 \supseteq E_3 \supseteq ...$ is a decreasing sequence of sets, with

$$E \equiv \lim_{n\to\infty} E_n \equiv \bigcap_{i=1}^{\infty} E_i$$

then

$$P(E) = P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$