M3/M4S3 STATISTICAL THEORY II

DIFFERENTIATION AND INTEGRATION

1. EXCHANGING THE ORDER OF DIFFERENTIATION AND INTEGRATION

Let $(\Omega, \mathcal{F}, \nu)$ be a general measure space, and for fixed $\theta \in \mathbb{R}$, let $f(\omega; \theta)$ be a Borel function on Ω. Suppose that

$$\frac{\partial f(\omega;\theta)}{\partial \theta}$$

exists almost everywhere for $\theta \in (a, b) \subset \mathbb{R}$, and that

$$\left| \frac{\partial f(\omega;\theta)}{\partial \theta} \right| \leq g(\omega) \qquad \text{a.e.}$$

for an integrable function g on Ω. Then for each $\theta \in (a, b)$, then $\dfrac{\partial f(\omega;\theta)}{\partial \theta}$ is integrable, and

$$\frac{d}{d\theta}\left\{ \int f(\omega;\theta)\, d\nu \right\} = \int \frac{\partial f(\omega;\theta)}{\partial \theta} d\nu$$

by the Lebesgue Dominated Convergence Theorem.

2. TRANSFORMATION/CHANGE OF VARIABLE

Let $(\Omega, \mathcal{F}, \nu)$ be a general measure space,and let f be a measurable function from $(\Omega, \mathcal{F}_\Omega)$ to $(\Lambda, \mathcal{F}_\Lambda)$. The **induced measure** by f is denoted by

$$\nu \circ f^{-1}$$

is a measure defined for $B \in \mathcal{F}_\Lambda$ by

$$\nu \circ f^{-1}(B) = \nu(f \in B) = \nu\left(f^{-1}(B)\right).$$

If g is a Borel function on $(\Lambda, \mathcal{F}_\Lambda)$, then

$$\int_\Omega (g \circ f)\ d\nu = \int_\Lambda g\, d\left(\nu \circ f^{-1}\right).$$

This is a change of variable formula for Lebesgue-Stieltjes integral.

3. PRODUCT SPACES AND PRODUCT MEASURE

Definition 1 *A measure ν on $(\Omega, \mathcal{F})$ is termed **sigma-finite ($\sigma-$finite)** if and only if there exists a sequence $\{A_i\}$ of sets in $\mathcal{F}$ such that*

$$\bigcup_{i=1}^{\infty} A_i \equiv \Omega$$

and $\nu(A_i) < \infty$ for all $i = 1, 2, 3, ...$

Theorem 1 *Let $(\Omega_i, \mathcal{F}_i, \nu_i)$ for $i = 1, 2$ be $\sigma-$finite measure spaces. Then for each $E \in \mathcal{F}_1 \times \mathcal{F}_2$ the function f_E defined on Ω_1 by*

$$f_E(\omega_1) = \nu_2(E_{\omega_1})$$

where

$$E_{\omega_1} \equiv \{\omega_2 : (\omega_1, \omega_2) \in E\}$$

for fixed ω_1 is ν_1-measurable. In addition, the set function ν defined on $\mathcal{F}_1 \times \mathcal{F}_2$ by

$$\nu(E) = \int_{\Omega_1} f_E \, d\nu_1 = \int_{\Omega_1} \nu_2(E_{\omega_1}) \, d\nu_1$$

is a σ-finite measure that is uniquely determined by the fact

$$\nu(A_1 \times A_2) = \nu_1(A_1)\,\nu_2(A_2)$$

for $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$.

Corollary. The function g_E defined on Ω_2 by

$$g_E(\omega_2) = \nu_1(E_{\omega_2})$$

where

$$E_{\omega_2} \equiv \{\omega_1 : (\omega_1, \omega_2) \in E\}$$

for fixed ω_2 is ν_2-measurable, and

$$\int_{\Omega_1} f_E \, d\nu_1 = \int_{\Omega_2} g_E \, d\nu_2.$$

Definition 2 ***Product Measure***
Let $(\Omega_i, \mathcal{F}_i, \nu_i)$ $i = 1, 2, ..., k$ be measure spaces with $\sigma-$finite measures. Then there exists a unique $\sigma-$finite measure on the product sigma-algebra

$$\sigma(\mathcal{F}_1 \times \mathcal{F}_2 \times ... \times \mathcal{F}_k)$$

*called the **product measure**. It is denoted*

$$\nu_1 \times \nu_2 \times ... \times \nu_k$$

and is defined by

$$\nu_1 \times \nu_2 \times ... \times \nu_k(A_1 \times A_2 \times ... \times A_k) = \prod_{i=1}^{k} \nu_i(A_i)$$

for all $A_i \in \mathcal{F}_i$, $i = 1, 2, ..., k$.

4. ITERATED AND DOUBLE INTEGRATION: FUBINI'S THEOREM

Let ν_i be a $\sigma-$finite measure on $(\Omega_i, \mathcal{F}_i)$ for $i = 1, 2$, and let f be a Borel function on

$$(\Omega_1, \mathcal{F}_1) \times (\Omega_2, \mathcal{F}_2)$$

whose integral with respect to product measure $\nu_1 \times \nu_2$ exists. For each $\omega_2 \in \Omega_2$, define function f_{ω_2} on Ω_1 by

$$f_{\omega_2}(\omega_1) = f(\omega_1, \omega_2) \qquad \omega_1 \in \Omega_1$$

with a similar definition for f_{ω_1} on Ω_2

$$f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2) \qquad \omega_2 \in \Omega_2.$$

(these functions are called **sections**). Then f_{ω_2} is ν_1-measurable, and f_{ω_1} is ν_2-measurable. If the two integrals

$$\int_{\Omega_1} f_{\omega_2}(\omega_1)\, d\nu_1 \qquad \text{and} \qquad \int_{\Omega_2} f_{\omega_1}(\omega_2)\, d\nu_2$$

exist for each ω_2 and ω_1 respectively, then functions α and β defined, respectively, by

$$\alpha(\omega_1) = \int_{\Omega_2} f_{\omega_1}(\omega_2)\, d\nu_2 \qquad \beta(\omega_2) = \int_{\Omega_1} f_{\omega_2}(\omega_1)\, d\nu_1$$

for $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ are measurable. If these functions are integrable wrt ν_1 and ν_2 respectively, then we denote the **iterated integral** of f by

$$\int_{\Omega_1} \left\{ \int_{\Omega_2} f_{\omega_1}(\omega_2)\, d\nu_2 \right\} d\nu_1 \equiv \int_{\Omega_1} \left\{ \int_{\Omega_2} f(\omega_1, \omega_2)\, d\nu_2 \right\} d\nu_1$$

which can also be denoted

$$\int \left\{ \int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2)\, d\nu_2 \right\} d\nu_1.$$

This is, in general, distinct from the **double integral** of f wrt the product measure

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2)\, d(\nu_1 \times \nu_2)$$

The next theorem gives conditions when the double integral is equal to the iterated integral.

Theorem 2 ***FUBINI'S THEOREM***
Let $(\Omega_i, \mathcal{F}_i, \nu_i)$ for $i = 1, 2$ be $\sigma-$finite measure spaces, and let f be a $\nu_1 \times \nu_2$-measurable function defined on $\Omega_1 \times \Omega_2$. Then

(a) If f is non-negative, then the functions α and β defined, respectively, on Ω_1 and Ω_2 by

$$\alpha(\omega_1) = \int_{\Omega_2} f_{\omega_1}(\omega_2)\, d\nu_2 \qquad \beta(\omega_2) = \int_{\Omega_1} f_{\omega_2}(\omega_1)\, d\nu_1$$

are measurable, and

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2)\, d(\nu_1 \times \nu_2) = \int_{\Omega_1} \left\{ \int_{\Omega_2} f(\omega_1, \omega_2)\, d\nu_2 \right\} d\nu_1 = \int_{\Omega_2} \left\{ \int_{\Omega_1} f(\omega_1, \omega_2)\, d\nu_1 \right\} d\nu_2 \qquad (1)$$

(b) If

$$\int_{\Omega_2}\left\{\int_{\Omega_1}|f(\omega_1,\omega_2)|\,d\nu_1\right\}d\nu_2$$

is finite, then f is integrable.

(c) If f is integrable, then almost every section of f is integrable, and the functions $\alpha(.)$ and $\beta(.)$ are integrable, and (1) holds.

Proof. (a) We establish

$$\int_{\Omega_1\times\Omega_2} f(\omega_1,\omega_2)\,d(\nu_1\times\nu_2)=\int_{\Omega_1}\left\{\int_{\Omega_2} f(\omega_1,\omega_2)\,d\nu_2\right\}d\nu_1$$

and deduce the rest of the result, as it is symmetric in indices 1 and 2. Suppose, initially, that $f=I_E$. Then

$$\beta(\omega_2)=\int_{\Omega_1}(I_E)_{\omega_2}\,d\nu_1=\nu(E_{\omega_2})$$

is a ν_2 measurable function, and by Theorem 1, equation (1) holds, and therefore it also holds for all simple functions, by the additivity of measures established previously. To prove the result for non-negative integrals, we use the Lebesgue Monotone Convergence Theorem. If f is a non-negative function, there is an increasing sequence, $\{\psi_n\}$, of simple functions which converges to f. Hence

$$\int_{\Omega_1\times\Omega_2} f(\omega_1,\omega_2)\,d(\nu_1\times\nu_2)=\lim_{n\to\infty}\int_{\Omega_1\times\Omega_2}\psi_n(\omega_1,\omega_2)\,d(\nu_1\times\nu_2).$$

Now, each section of a simple function is simple, and also $\lim_{n\to\infty}(\psi_n)_{\omega_2}=f_{\omega_2}$. Thus the function

$$\beta_n(\omega_2)=\int_{\Omega_1}(\psi_n)_{\omega_2}\,d\nu_1 \qquad n=1,2,...$$

defines an increasing sequence of non-negative measurable functions with

$$\lim_{n\to\infty}\beta_n=\beta \qquad \text{where} \qquad \beta(\omega_2)=\int_{\Omega_1} f_{\omega_2}(\omega_1)\,d\nu_1.$$

and hence β is measurable with

$$\int_{\Omega_2}\left\{\int_{\Omega_1} f(\omega_1,\omega_2)\,d\nu_1\right\}d\nu_2=\int_{\Omega_2}\beta(\omega_2)\,d\nu_2=\lim_{n\to\infty}\int_{\Omega_2}\beta_n(\omega_2)\,d\nu_2=\lim_{n\to\infty}\int_{\Omega_2}\left\{\int_{\Omega_1}\psi_n d\nu_1\right\}d\nu_2$$

and this proves the result.

(b) This result follows from (a) applied to the function $|f|$.

(c) This result follows as if f is integrable, then so are the positive and negative part functions f^+ and f^-, and thus by (a) these non-negative functions are integrable, and the iterated integrals are finite. Thus the iterated integral(s) of f is finite. Finally, the function β defined above is finite a.e., since its integral with respect to ν_2 is finite (this integral is merely one of the parts of the iterated integral). Thus f_{ω_2} (and by symmetry of argument, f_{ω_1}) is integrable a.e. for all ω_2 (as is f_{ω_1} for any ω_1). ∎