M3/M4S3 STATISTICAL THEORY II

COURSE SUMMARY

1. ASYMPTOTIC THEORY

1.1 Basic Mathematical Tools and Notation

- limits, order notation
- continuity, limits of functions
- supremum, infimum
- limit superior, limit inferior

1.2 Probability Spaces

- Sigma Algebras
- Measure Spaces
- Borel sets in $\mathbb{R}$
- Measure and its properties : continuity of measure
- Measurable functions
- Indicator Functions
- *Representation Theorem using Simple Functions*
- Properties of Borel functions

1.3 Integration

- Integral with respect to measure: Lebesgue-Stieltjes Integration
- Simple Functions and their integrals
- Integrals of Borel functions: The Supremum Definition, existence and integrability
- Sets of measure zero and almost sure/almost everywhere existence
- Basic Properties of integrals with respect to measure

1.4 Theorems and Results for Lebesgue-Stieltjes Integration

- *Lebesgue Monotone Convergence (***Proof not examinable***)*
- *Fatou's Lemma (***Proof not examinable***)*
- *Lebesgue Dominated Convergence (***Proof not examinable***)*
- Product Measure and *Fubini's Theorem (***Proof not examinable***)*
- Application to expectation calculations

1.5 **Convergence for Random Variables**

- Convergence in Law
- Convergence in Probability
- Convergence in r^{th} mean
- Convergence almost surely
- *Theorem: Equivalence of almost sure convergence definitions*
- *Theorem: Relations between the modes of convergence*
- *Theorem: Partial Converses (eg Scheffe's Theorem)*
- *Theorem: The Borel-Cantelli Lemma*

1.6 **Laws of Large Numbers**

- Characteristic Functions and their properties
- Mean-Value Theorem and Taylor Theorem
- *Theorem: The Weak and Strong Laws*
- Consistency
- The empirical distribution function
- *Theorem: The Glivenko Cantelli Lemma*

1.7 **Central Limit Theorems**

- *Theorem: The basic (Lindeberg-Levy) CLT*
- Cramer-Wold device
- Helly-Bray Theorem
- Continuity Theorem
- Asymptotic Normality
- Berry-Esseen Result
- *Theorem: The Lindeberg-Feller CLT for non-iid case (statement not proof)*
- *Slutsky Theorems*
- *Cramer's Theorem (for transformed variables)*
- The Chi-square statistic and its distribution
- *Theorem: Asymptotic Distribution of Sample Quantiles*

2. **Likelihood Theory and Extensions**

2.1 **Extending the Strong Law**

- The Le Cam Results on Uniform Strong Consistency

2.2 **Maximum Likelihood Estimation**

- Basic MLE approach
- Solutions to the Likelihood Equations
- Kullback-Liebler divergence
- *Theorem: Properties of the KL divergence*
- *Theorem: Wald Theorem on the Consistency of the MLE (Proof assuming Le Cam (B))*
- Efficient Estimation: Score Function and Fisher Information

2.3 **Asymptotic Normality of the MLE**

- *Theorem: Cramer Theorem on the Asymptotic Normality of the MLE (Proof assuming Le Cam/Wald Theorems)*

2.4 **The Cramer-Rao Bound**

- *Theorem: The Information Inequality and the Cramer-Rao bound*
- Nuisance Parameters

2.5 **Likelihood-Based Hypothesis Tests**

- Likelihood-Ratio Statistic
- Wald Statistic
- Rao/Score Statistic
- Asymptotic Distributions under the null hypothesis
- Composite Null Hypotheses and Nuisance Parameters

2.6 **Modified Likelihoods**

- Partitioning the Information matrix in the presence of nuisance parameters
- Orthogonality
- Profile Likelihood
- Approximation formulae
- Connections with likelihood-ratio tests
- Modified Profile Likelihood
- Marginal and Conditional Likelihood
- Quasi-Likelihood

3. **Bayesian Theory**

3.1 **Simple Bayesian Computations**

3.2 **Representation Theorems**

- *Theorem: The 0-1 representation theorem*
- The General Representation Theorem

3.3 **Asymptotic Normality of the Posterior Distribution**