M3S3/M4S3
ASSESSED COURSEWORK 1

SOLUTIONS

1.(a) We have $E[X_{nj}] = 0, Var[X_{nj}] = E\left[X_{nj}^2\right] = 1$, for all n and j, so $v_n^2 = n$. Now, for fixed $\varepsilon > 0$

$$E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] = \begin{cases} 0 & \text{if } 1 < \varepsilon\sqrt{n} \\ n & \text{if } 1 \geq \varepsilon\sqrt{n} \end{cases} = \begin{cases} 0 & \text{if } 1/\sqrt{n} < \varepsilon \\ n & \text{if } 1/\sqrt{n} \geq \varepsilon \end{cases}$$

so that

$$\sum_{j=1}^{n} E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] = \begin{cases} 0 & \text{if } 1/\sqrt{n} < \varepsilon \\ n^2 & \text{if } 1/\sqrt{n} \geq \varepsilon \end{cases}$$

and

$$\frac{1}{v_n^2} \sum_{j=1}^{n} E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] = \begin{cases} 0 & \text{if } 1/\sqrt{n} < \varepsilon \\ n & \text{if } 1/\sqrt{n} \geq \varepsilon \end{cases}.$$

Thus, in the limit as $n \to \infty$, $1/\sqrt{n} \to 0$, so

$$\frac{1}{v_n^2} \sum_{j=1}^{n} E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] \to 0$$

as required for the Lindeberg condition to be met. Hence

$$\frac{T_n}{\sqrt{n}} \xrightarrow{\mathfrak{L}} Z \sim N(0,1) \qquad \text{and} \qquad T_n \sim AN(0,n)$$

[4 MARKS]

(b) We have $E[X_{nj}] = 0, Var[X_{nj}] = E\left[X_{nj}^2\right] = j^2$, so

$$v_n^2 = \sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1) \tag{1}$$

Note that $v_n^2 = O\left(n^3\right)$ Now, for fixed $\varepsilon > 0$

$$E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] = \begin{cases} 0 & \text{if } j < \varepsilon v_n \\ j^2 & \text{if } j \geq \varepsilon v_n \end{cases} = \begin{cases} 0 & \text{if } j/v_n < \varepsilon \\ j^2 & \text{if } j/v_n \geq \varepsilon \end{cases}$$

so that, taking the smallest possible j in the strict inequality

$$\sum_{j=1}^{n} E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] = \begin{cases} 0 & \text{if } 1/v_n < \varepsilon \\ s_n(\varepsilon) & \text{if } 1/v_n \geq \varepsilon \end{cases}$$

where $0 < s_n(\varepsilon) \leq v_n^2$ and $s_n(\varepsilon)$ is given by the sum in (1) suitably truncated, unless $n \geq \varepsilon v_n$.

$$\frac{1}{v_n^2} \sum_{j=1}^{n} E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] = \begin{cases} 0 & \text{if } 1/v_n < \varepsilon \\ s_n/v_n & \text{if } 1/v_n \geq \varepsilon \end{cases}.$$

Thus, in the limit as $n \to \infty$, $1/v_n \to 0$, so

$$\frac{1}{v_n^2} \sum_{j=1}^{n} E\left[X_{nj}^2 I_{\{|X_{nj}| \geq \varepsilon v_n\}}\right] \to 0$$

as required for the Lindeberg condition to be met. Hence

$$\frac{T_n}{\sqrt{\frac{1}{6}n(n+1)(2n+1)}} \xrightarrow{\mathfrak{L}} Z \sim N(0,1) \qquad \text{and} \qquad T_n \sim AN\left(0, \frac{n(n+1)(2n+1)}{6}\right)$$

[6 MARKS]

2. (i) We have from the standard Central Limit Theorem that

$$\sqrt{n}\left(\overline{X}_n - \lambda\right) \xrightarrow{\mathfrak{L}} Z \sim N(0, \lambda)$$

as $E[X_i] = Var[X_i] = \lambda$. Hence

$$\overline{X}_n \sim AN\left(\lambda, \frac{\lambda}{n}\right)$$

as $n \to \infty$. Let $g(x) = xe^{-x}$, so that

$$\dot{g}(x) = e^{-x}(1-x)$$

which is continuous on $\mathbb{R}$ and thus by Cramer's Theorem in the univariate setting

$$\sqrt{n}\left(g\left(\overline{X}_n\right) - g(\lambda)\right) \xrightarrow{\mathfrak{L}} Z_\lambda \sim N\left(0, \{\dot{g}(\lambda)\}^2 \lambda\right)$$

that is

$$\sqrt{n}\left(Y_n - \lambda e^{-\lambda}\right) \xrightarrow{\mathfrak{L}} Z_\lambda \sim N\left(0, e^{-2\lambda}(1-\lambda)^2 \lambda\right)$$

and

$$Y_n \sim AN\left(\lambda e^{-\lambda}, \frac{e^{-2\lambda}(1-\lambda)^2 \lambda}{n}\right).$$

When $\lambda = 1$, this yields only

$$Y_n \xrightarrow{\mathfrak{L}} \lambda e^{-\lambda}$$

which is correct, but not useful as an asymptotic distribution.

[5 MARKS]

(ii) In Procedure A, we have by the Central Limit Theorem

$$\sqrt{n}\left(\frac{X}{n}-\theta\right) \xrightarrow{\mathfrak{L}} Z_\theta \sim N\left(0, \theta\left(1-\theta\right)\right)$$

and thus

$$\widehat{\theta}_A = \frac{X}{n} \sim AN\left(\theta, \frac{\theta\left(1-\theta\right)}{n}\right).$$

Similarly, for Procedure B

$$\sqrt{n}\left(\frac{Y}{n}-\theta^2\right) \xrightarrow{\mathfrak{L}} Z_\theta \sim N\left(0, \theta^2\left(1-\theta^2\right)\right).$$

Let $\phi = \theta^2$. Now, setting $g(x) = \sqrt{x}$, we have

$$\dot{g}\left(x\right) = \frac{1}{2\sqrt{x}}$$

which is continuous when $x \neq 0$, and thus by Cramer's Theorem

$$\sqrt{n}\left(g\left(\frac{Y}{n}\right) - g\left(\phi\right)\right) \xrightarrow{\mathfrak{L}} Z_\theta \sim N\left(0, \phi\left(1-\phi\right)\left\{\dot{g}\left(\phi\right)\right\}^2\right)$$

or

$$\sqrt{n}\left(\sqrt{\frac{Y}{n}} - \sqrt{\phi}\right) = \sqrt{n}\left(\sqrt{\frac{Y}{n}} - \theta\right) \xrightarrow{\mathfrak{L}} Z_\theta \sim N\left(0, \frac{\phi\left(1-\phi\right)}{4\phi}\right) = N\left(0, \frac{\left(1-\phi\right)}{4}\right)$$

yielding

$$\widehat{\theta}_B = \sqrt{\frac{Y}{n}} \sim AN\left(\theta, \frac{\left(1-\theta^2\right)}{4n}\right)$$

Thus the asymptotic variance is smaller for procedure A when

$$\frac{\theta\left(1-\theta\right)}{n} < \frac{\left(1-\theta^2\right)}{4n}$$

that is, when

$$4\theta < \left(1+\theta\right).$$

or

$$3\theta < 1.$$

This holds for θ where $0 < \theta < \dfrac{1}{3}$.

[5 MARKS]